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Some constraints on inflation models with power-law potentials

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We investigate inflation in a closed Friedmann-Robertson-Walker universe filled with the scalar field with a power-law potential. For a wide range of powers and parameters of the potential we numerically calculate the slow-roll parameters and scalar spectral index at the epoch when the present Hubble scale leaves the horizon and at the end of inflation. Also we compare the results of our numerical calculations with recent observational data. This allows us to set a constraint on the power of the potential: $\alpha \lesssim (3.5–4.5)$.

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I. INTRODUCTION

Inflation, the stage of accelerated expansion of the Universe, first proposed in the beginning of the 1980s [1], nowadays receives a great deal of attention as well. In our previous paper [2] we investigated the generality of inflation for a wide class of quintessence potentials and noted that the criteria we used is insufficient to decide about the inflationarity of the model with a given potential. So in this paper we proceed with the investigation of closed Friedmann-Robertson-Walker (FRW) models, but using power-law potentials only. In this way we calculate the spectral indices, slow-roll parameters, and other values on the epoch, when the present Hubble scale leaves the horizon and at the end of inflation, to compare our constraints with earlier obtained results and with observation data. This allows us to make some constraints on FRW models with a scalar field, on potentials of the scalar field and on parameters of these potentials. Since, due to inflation, the Universe becomes flat very quickly our constraints are applicable to the flat case as well. If one supposes that the same potential describes both the inflationary stage in the early Universe and acceleration nowadays we can compare our constraints with constraints on the quintessence potentials (see e.g. [3,4] or [5–7] for review of the problem).

Another aim of this paper is the investigation of the influence of the initial conditions on inflation dynamics in the case of closed FRW models. Really, in the case of an initially flat universe we have only one degree of freedom—how does primordial energy density distribute between kinetic and potential terms? But in the case of the initially closed Universe we have at least one more degree of freedom—the distribution between curvature and initial expansion rate terms. So our second aim is in studying these two distributions, comparing them to each other, and investigating how they act on inflation.

II. MAIN EQUATIONS

The equations describing the evolution of the Universe in a closed FRW model are

$$\frac{m_P^2}{16\pi}\left(\ddot{a}+\frac{\dot{a}^2}{2a}+\frac{1}{2a}\right)+\frac{a}{4}\left(\frac{\dot{\varphi}^2}{2}-V(\varphi)\right)=0,$$

$$\ddot{\varphi}+\frac{3\dot{\varphi}\dot{a}}{a}+\frac{dV(\varphi)}{d\varphi}=0,$$

and the first integral of the system is

$$\frac{3m_P^2}{8\pi}\left(\frac{\dot{a}^2}{a^2}+\frac{1}{a^2}\right)=\left(V(\varphi)+\frac{\dot{\varphi}^2}{2}\right),$$

where $m_P=1/\sqrt{G}=1.2\times 10^{19}$ GeV.

And we use the *trigonometrical* (*angular*) parametrization (ϕ,H_0) of the space of initial conditions:

$$\frac{3m_P^2}{8\pi}\left(H_0^2+\frac{1}{a^2}\right)=m_P^4,\quad H_0^2+\frac{1}{a^2}=\frac{8\pi m_P^2}{3}, \tag{1}$$

$$H_0\in\left[0;\sqrt{\frac{8\pi}{3}}m_P\right),$$

$$V(\varphi)+\frac{\dot{\varphi}^2}{2}=m_P^4,\quad V(\varphi)=m_P^4\cos^2\phi,$$

$$\frac{\dot{\varphi}^2}{2}=m_P^4\sin^2\phi,\quad \phi\in\left[-\frac{\pi}{2};\frac{\pi}{2}\right].$$

Our method is as follows. As in [2] we start from the Planck boundary for a given pair of initial conditions (ϕ,H_0) and then numerically calculate the further evolution of the Universe through inflation. Also we calculate scalar spectral index and slow-roll parameters during the evolution of the Universe. And since the Universe becomes flat very quickly due to inflation, we can use the usual determination for scalar spectral index. There are two expressions for it—first order [8] and second order [9]—and we use second order results for more precise calculations. Also one can express them using *potential slow-roll approximation* (PSRA) [8,10] and *Hubble slow-roll approximation* (HSRA) [11] (see [12] for details). Equations for slow-roll parameters are

for PSRA

$$\epsilon_V(\varphi)=\frac{m_P^2}{16\pi}\left(\frac{V'(\varphi)}{V(\varphi)}\right)^2,$$

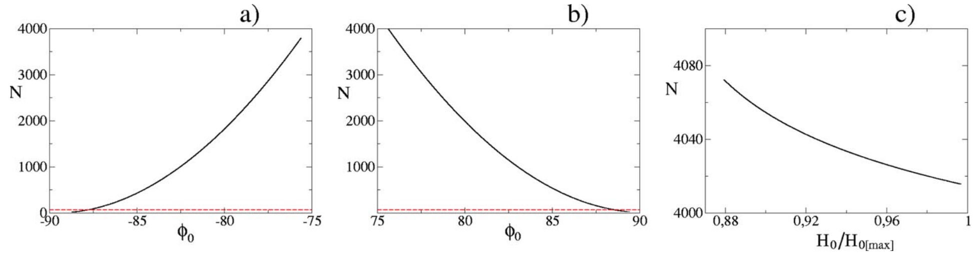

FIG. 1. The dependence of the number of *e*-foldings a universe experienced during inflation on the initial conditions for the angular parametrization. It is clear that the dependence on ϕ [(a) for the case of positive ϕ and (b) for the case of negative ϕ] is stronger that the dependence on H (c) (see text for details).

$$\eta_V(\varphi)=\frac{m_P^2}{8\pi}\frac{V''(\varphi)}{V(\varphi)},$$

for HSRA

$$\epsilon_H(\varphi)=\frac{m_P^2}{4\pi}\left(\frac{H'(\varphi)}{H(\varphi)}\right)^2, \quad \eta_H(\varphi)=\frac{\hat{m}_P^2}{4\pi}\frac{H''(\varphi)}{H(\varphi)},$$

and for n_s, respectively,

$$n_s=1-6\epsilon_V+2\eta_V+\frac{1}{3}(44-18c)\epsilon_V^2+(4c-14)\epsilon_V\eta_V+\frac{2}{3}\eta_V^2+\cdots, \tag{2}$$

$$n_s=1-4\epsilon_H+2\eta_H-2(1+c)\epsilon_H^2-\frac{1}{2}(3-5c)\epsilon_H\eta_H-\frac{1}{2}(3-c)\eta_H^2+\cdots, \tag{3}$$

where $c\equiv 4(\ln 2+b)-5\simeq 0.08145$ with Euler-Mascheroni constant b. So we calculate n_s by both these cases and compare them to each other. Also we calculate slow-roll parameters and compare them at the epoch when the present Hubble scale leaves the horizon for different potentials and for different parameters of the potentials.

Since the purpose of our paper is to set constraints on inflation models, we compare results of our numerical calculations with constraints on n_s and other values from experiments. In particular we compare them with results obtained in Ref. [3]; so we need to introduce horizon-flow parameters ϵ_1, ϵ_2 (see [13,14] for more details):

$$\epsilon_1\simeq\frac{m_P^2}{16\pi}\left(\frac{V'}{V}\right)^2, \tag{4}$$

$$\epsilon_2\simeq\frac{m_P^2}{4\pi}\left[\left(\frac{V'}{V}\right)^2-\frac{V''}{V}\right]. \tag{5}$$

To compare our constraints with [3] we need to localize the moment during inflation when the present Hubble scale leaves the horizon. It took place at about 60 *e*-folds before the end of inflation ([15], see also [16] for details). This value is model-dependent and it depends also on the way inflation ends so to simplify we use two values for this: 62 as a bound value and 50. And so we calculate all parameters on these two epochs: 62 and 50 *e*-folds before the end of inflation. For further use we denote this value as $N_{\rm hor}$.

III. POWER-LAW POTENTIAL

First, let us describe the dependence of the number of *e*-foldings the Universe experienced during inflation on initial conditions. To illustrate this we plotted in Fig. 1 this dependence on ϕ and on H separately; to realize the whole picture one needs to multiply these functions. So in Fig. 1(a) there is a dependence of the number of *e*-foldings the Universe experienced during inflation on ϕ for negative ϕ, in Fig. 1(b) the same but for positive ϕ, and in Fig. 1(c), on H_0. From Figs. 1(a) and 1(b) one can also see the influence of the sign of initial $\dot{\varphi}$—positive ϕ corresponds to positive initial $\dot{\varphi}$ and negative ϕ corresponds to negative initial $\dot{\varphi}$. For instance, for $H_0=H_{0[max]}$, the flat case, measure of trajectories experienced insufficient inflation (the case when the Universe experienced inflation but the number of *e*-foldings is less than 70) is about 2.26° for negative ϕ and about 1.49° for positive ϕ (so for the flat case the Universe experienced sufficient inflation for $-87.75°<\phi<88.51°$; in Figs. 1(a) and 1(b) the dashed line corresponds namely to $N=70$ *e*-foldings).

From Fig. 1 one can also learn that the dependence of the number of *e*-foldings on H_0 is many times weaker than the dependence on ϕ. Really, ϕ determines the initial distribution of the energy between kinetic and potential terms and, since the Universe very quickly reaches the slow-roll regime, it determines the energy density at the beginning of inflation. Also during reaching the slow-roll regime H becomes large, so H_0—the initial value of H acts weakly on the energy density at the beginning of inflation.

Power-law potentials are well studied and they lead to "chaotic inflation" [17]. One can really use them as the inflation part in potentials like those considered by Peebles and Vilenkin [18]. They have also attracted attention for some of their properties [19,20].

We consider power-law potentials of a kind

$$V(\varphi)=\frac{\lambda\varphi^\alpha}{\alpha}=\lambda*\left(\frac{\varphi}{m_P}\right)^\alpha, \tag{6}$$

and our results are as follows. In Fig. 2 we plotted positions of the models with different powers on the (ϵ_1,ϵ_2) plane in

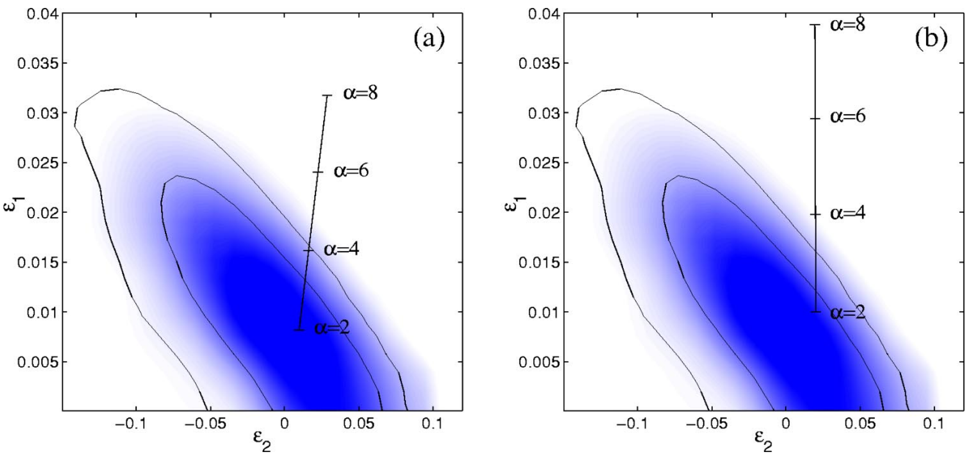

FIG. 2. Positions of models with different powers of power-law potentials on (ϵ_1,ϵ_2) plane in the case of $N_{\rm hor}=62$ at (a) panel and $N_{\rm hor}=50$ at (b) panel.

the case of $N_{\rm hor}=62$ at (a) panel and in case of $N_{\rm hor}=50$ at (b) panel. By comparison these plots with bounds on (ϵ_1,ϵ_2) plane obtained in [3] from 2dF and WMAP data one can make a constraint on α as $\alpha\lesssim 4.8$ in the case of $N_{\rm hor}=62$ and $\alpha\lesssim 3.5$ in the case of $N_{\rm hor}=50$.

Another constraint on power α can be obtained from Fig. 3. In Fig. 3 we plotted the dependence of n_s: calculated in PSRA [see Eq. (2)] we denoted as ${}^{[V]}n_s$ and calculated in HSRA [see Eq. (3)] we denoted as ${}^{[H]}n_s$. Using bounds on n_s [21]: $n_s=0.99\pm0.04$ (WMAP only) and $n_s=0.97\pm0.03$ (WMAP+ACBAR+CBI+2dFGRS+L_α-forest) one can set a bound $\alpha\lesssim 4.5$ in case of $N_{\rm hor}=62$ and $\alpha\lesssim 3.8$ in case of $N_{\rm hor}=50$. One can see these two constraints—from n_s and the previous one—are close to each other.

Now let us set a constraint on λ^*. To do this we can use results obtained from COBE data in [14]:

$$3<\frac{V_{\rm infl}^{1/4}}{10^{15}\ \rm GeV}<29, \qquad (7)$$

and we calculate these values at the end of inflation. After defining $K(\alpha)=(\varphi_{\rm end}/m_P)$ one can obtain $V_{\rm end}=\lambda^* K^\alpha(\alpha)$ and after substitution of Eq. (6) to Eq. (7):

$$\lambda_2^*<\lambda^*<\lambda_1^*,$$

where

$$\lambda_1^*=3.4\times10^{-11}m_P^4K^{-\alpha}(\alpha),$$

$$\lambda_2^*=3.9\times10^{-15}m_P^4K^{-\alpha}(\alpha).$$

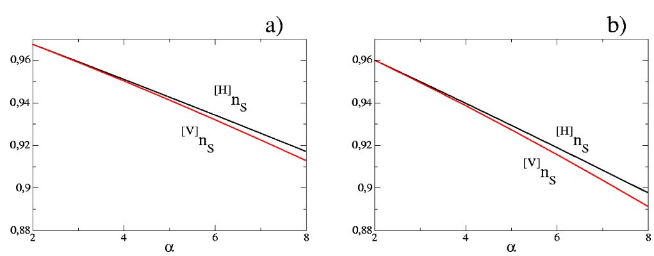

FIG. 3. The dependence of n_s, calculated in PSRA and HSRA (see text for details), on power α in the case of $N_{\rm hor}=62$ at (a) panel and in the case of $N_{\rm hor}=50$ at (b) panel.

One needs to keep in mind the relation between λ and λ^* to recalculate λ^* into λ and inversely:

$$\lambda=\frac{\alpha\lambda^*}{m_P^\alpha},\qquad \lambda^*=\frac{\lambda m_P^\alpha}{\alpha}.$$

Another test, also linked with COBE normalization [22], is about the density perturbation spectrum $A_S(k)$:

$$A_S^2(k)=\frac{512\pi}{75}\left.\frac{V^3}{m_P^6V'^2}\right|_{k=aH}, \qquad (8)$$

and this value is calculated on the epoch when the present Hubble scale leaves the horizon. Also we can define the value of the field on this epoch as $\varphi_{\rm hor}=L(\alpha)m_P$ and so rewrite Eq. (8) using Eq. (6) as

$$A_S^2=\frac{512\pi}{75m_P^4}\frac{\lambda^*L^{\alpha+2}(\alpha)}{\alpha^2}.$$

Let us remind the reader that according to COBE data this value is about $A_S\approx2\times10^{-5}$. Using it one can get another estimation for λ^*:

$$\lambda_{A_S}^*=\frac{3\times10^{-8}m_P^4\alpha^2}{512\pi L^{\alpha+2}(\alpha)}.$$

Finally, in Fig. 4 we plotted the dependence of both λ_1^* and λ_2^* on α and the dependence of $\lambda_{A_S}^*$ on α. The range of possible values of λ^* due to Eq. (7) [14] is between curves λ_1^* and λ_2^*. We plotted $\lambda_{A_S}^*$ for cases $N_{\rm hor}=62$ and $N_{\rm hor}=50$. One can see from Fig. 4 that in the case of $N_{\rm hor}=62$ we have a constraint $\alpha\lesssim3.5$ and in the case of $N_{\rm hor}=50$ we have a constraint $\alpha\lesssim4.0$.

IV. CONCLUSIONS

So we reached our aim—we set some constraints on power-law potentials and their parameters. We calculated the whole evolution of the Universe during inflation for a wide range of initial conditions, parameters of power-law potentials (power α and λ), and set some constraints on power-law potentials and on their parameters. Also we compared our constraints with results obtained from the recent cosmic

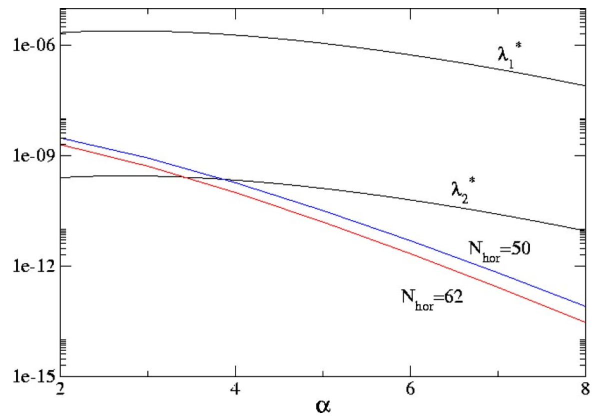

FIG. 4. The dependence of λ_1^* and λ_2^* as well as $\lambda_{A_S}^*$, calculated for $N_{\rm hor}=62$ and $N_{\rm hor}=50$, on the power α (see text for details).

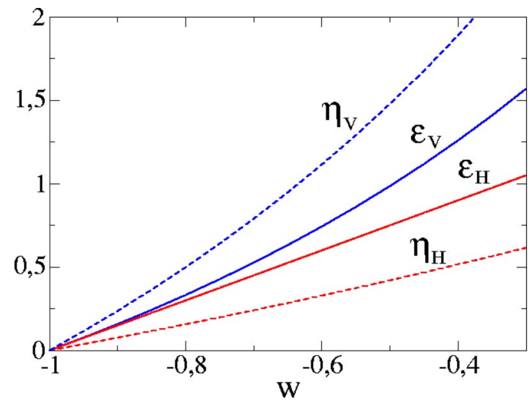

FIG. 5. An example of the behavior of slow-roll parameters with respect to the equation of state of the scalar field near the end of inflation. It is clear that at the moment when inflation ends ($w=-1/3$) only ϵ_H is equal to unity but other parameters are not (see text for details).

microwave background (CMB) data and large scale structure (LSS) data [3,16,15]. As we noted above for an epoch when the present Hubble scale leaves the horizon we have used two values—62 *e*-folds before inflation ends and 50 *e*-folds. And 62 *e*-folds is bound value in the sense that other possible values are smaller than 62. We used 50 *e*-folds namely as an example of such a value and to demonstrate what can happen with n_s, ϵ_1 and other values if we use a lower (than 62) number of *e*-folds before the end of inflation to determine the epoch when the present Hubble scale leaves the horizon. The result we obtained is $\alpha \lesssim (3.5\text{–}4.5)$. The exact value is very model dependent. It depends on many factors: first on the way inflation ends—this determines the epoch when the present Hubble scale leaves the horizon. From figures one can see how it acts on the results. Also it depends on the observation data—to make our constraints more precise we need more precise observation data. But even with these uncertainties our constraints are harder than some results obtained from the CMB and LSS data [3,15,16].

One can see that our numerical results differ somewhat from analytical results. This is due to the fact that one can incorrectly determine the exact moment when inflation ends using the relation $\max\{\epsilon,\eta\}=1$. As one can see from Fig. 5 (we plotted it only as an example; it corresponds to the case $\alpha=4$) at the *real* moment of the end of inflation only ϵ_H is exactly equal to unity. And since most analytical results are obtained using relation $\epsilon_V=1$, it does not correspond to the exact moment of the end of inflation. The small difference between our numerical results and analytical results is namely due to this uncertainty.

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